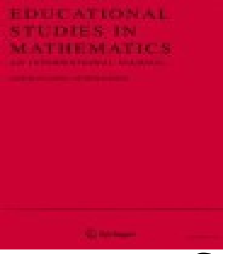

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Conceptual Understanding of Limits and Continuity of Functions: Senior four Rwandan Secondary Schools.

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Abstract.

The aim of this study is to examine the student's conceptual understanding of limits and continuity of functions of one real variable, in upper secondary school in two districts of Rwanda. The study employed a mixed methods research approach which is explanatory sequential research design. The sample consisted of 21 teachers and 252 senior four students with mathematics in their option. The data sources included achievement tests, classroom observations, focus group interviews, and informal discussions before and after classroom instruction. After analyzing students' errors committed when solving limits and continuity functions, the findings showed that most of the learners indicated different categories of misconceptions and errors in limits and continuity. The source of these errors was ~~probably~~ the methodology used to teach and teachers' lack of conceptual understanding of these functions. The study recommends that teachers should be empowered with skills to help their students handle misconceptions about limits and continuities.

Keywords: Calculus, conceptual understanding, limits and continuity concepts, mathematical errors, Teaching mathematics.

Introduction

The concept of limit has been confused with vague and sometimes philosophical notions of *infinity*, such as infinitely large and infinitely small numbers. Other mathematical entities were also confused with subjective and undefined geometric intuitions (Denbel, 2014).The concept of limits was invented to solve three types of difficulties related to the calculation of areas of geometric figures, the elimination of geometric lengths, and consideration of the geometric nature of lengths. Additionally, several other concepts are related to the limit concept, including functions and infinitesimals, sum and convergence of a series and differentiation. Students who understand limits are more likely to understand the concepts connected to them, but students often struggle to understand it (Juter, 2006). The aim of its development was to give differential and integral calculus a rigorous foundation. The concept of limit was only used correctly and carefully in exceptional cases until this time. The problem of handling limits has even proven challenging for many great mathematicians (Edwards et al., 2005).

A review of the curriculum, textbooks, exercises, and examinations reveals that certain aspects of the limit re given greater emphasis in mathematics teaching. During the first half of the twentieth century in 1946, French mathematics textsbooks introduced the notion of derivatives intuitively without providing a formal definition of limit (Boyer, 1959). A proper introduction of the concept was made in 1966 where limits was typically introduced in mathematics curriculum by including a formal definition, a statement of their uniqueness, and general arithmetic theorems (Cornu & Tall, 1991). Besides~~,~~ that before 2015, in the old Rwandan curriculum which was knowledge-based curriculum (KBC), the concept of limits was taught in senior five. During the review of the KBC to competence based curriculum (CBC) ,the concept of limits was greatly emphasized and is taught from ~~grades~~ senior four to senior six as the key concept of calculus and mathematical analysis (REB, 2015). The students need to master the concept of limit and continuity in senior four. This helps them understand mathematical analysis which depends on fundamental principle in approximation, differential and integral calculus and further mathematics at college or university level. According to Sofronas (2011), majority of the first-year university students are unable to link concepts to skills, and neither mastered calculus concepts nor fundamental skills. Furthermore, due to that skills gap, advanced calculus topics and other related subjects are difficult for students to understand. Additionally, Muzangwa and Chifamba (2012) found that knowledge gaps in basic algebra were to blame for most errors in learning calculus. Students' choice of strategies in solving mathematics problems was affected by their understanding of basic concepts. Also, there have been fewer studies on learners' ability to interpret informal and precise definitions that communicate clearly and naturally the concept of limits (Ashraf, 2020; Voon et al., 2017).

Although mathematics plays a significant role in various aspects of life, there is a negative attitude towards mathematics and sciences which leads to low performance (de Vera et al., 2022; Kunwar, 2021; Mbonyiryivuze et al., 2021). Poor performance in secondary school mathematics has persisted in the recent past (Thorpe, 2018). One of the reasons is that some mathematics topics are perceived to be difficult for students and students present some misconceptions (Bridgers, 2007). For instance, calculus concepts are complex in their transition from infinitesimal to formal forms and students find some difficulties to understand them

(Bressoud et al., 2017). This means that students are challenged with understanding the concepts of limits and functions. Therefore, instructors need to know the common misconceptions related to calculus, such as those related to functions, continuity, sequences, limits, derivatives, and integrals which leads to lower performance in mathematics (Galanti & Miller, 2021). Although students' challenges in calculus have been explored, the situation in senior secondary school students in Rwanda remains underexplored. Therefore, this study seeks to explore secondary school students' conceptual understanding of limits and continuity functions. Specifically, the study seeks to answer the questions:

1. What learning difficulties do secondary school students present in lessons on limits and continuity function?
2. What misconceptions and errors do secondary school students present in lessons on limits and continuity function?
3. What are the factors contributing to learners' misconceptions of limits and continuities?

Literature Review

Contribution of calculus in other fields

Calculus is branch of mathematics that studies continuous change. It is divided into differential and integral calculus. Differential calculus focuses on the instantaneous rate of change, while integral calculus analyzes quantities and areas under curves (Stewart, 2015). Calculus is applied in Physics, Engineering, Economics, Agriculture, Medicine, and Biology. In Mechanics as a branch of physics to find velocity, displacement requires Calculus concept. Calculating bacterial growth rates based on temperature and food source is done using differential calculus in Biology. In Medicine, cardiac output and blood flow are measured with the help of calculus in medical tests. Economists use calculus to determine marginal revenues and cost. In Agriculture, temperature, humidity, and pressure changes are predicted by Differential Equations based on varying weather conditions (Anton et al., 2021)).

For beneficial of learner's academic success, calculus is also incredibly important for their future careers. We are often curious to know how a change in one quantity affects another related quantity in our everyday lives. It is the reason that calculus is taught from high school up to the university level.

Concept of limits and continuity of function in one variable analysis

Limits are basic concepts in calculus that describe how a function behaves near a specific input (Bressoud et al., 2017). In mathematics, limits are studied for the purpose of understanding continuities, differentiability, asymptotes of function, convergence of improper integrals and series. Many different techniques exist for evaluating the limits of functions, including using definition, recognizing a pattern, simple substitution, algebraic simplification, and graphical methods. Mathematically, x may approach value a from above (right) or below (left), in which case the limits may be written as $\lim_{x \to a^+} f(x)$ and $\lim_{x \to a^-} f(x)$ respectively. The function admits a limit at a point a if and only the left and right limit are equal.

Consequently, $f(x)$ is continuous at a if and only if

i. $f(x)$ is defined at a

ii. $\lim_{x \to a} f(x) = f(a)$

A discontinuity can be removable or non-removable. Non-removable discontinuities may either be jumps or infinite discontinuities. A removable discontinuity occurs when a hole appears in the graph of a function. Even if the hole is filled in at one end, a jump discontinuity occurs when a limit of function does not exist or on a graph, two ends of functions do not meet. A discontinuity that spikes up to infinity at both ends refers to an infinite discontinuity or both sides of a function reaching an asymptote,

i.e. $\lim_{x\to a} f(x) = \pm\infty$.

The ideas of limits and continuity have drawn a lot of attention in studies where different approaches students take to think about these subjects from theoretical to practical have been noted (Areaya & Sidelil, 2012). Students need to be able to understand three types of rationality to understand limits and continuity where observations are not generalized in pragmatic rationality; only specific cases are examined; to deduce generalizations, empirical rationality is applied; and theories, properties, lemma and corollary are established based on theories which describe theoretical rationality(Branchetti et al., 2020)

Concept of limits on the approach of Cauchy and Heine's approach

There are several reasons why teaching the Epsilon-Delta definition of limit is difficult for anyone who has attempted to do so. Teachers should find an approach that could allow students to justify and prove their ideas rather than memorizing without understanding.

Definition of Limits of function using Cauchy

Assume $f(x)$ be a function defined on an open interval around a. We say that the limit of $f(x)$ as tends a is L. i.e. L if $\forall\varepsilon > 0, \exists\delta > 0 : \; 0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon$. Geometrically,$x \in \,]a - \delta, a + \delta[\Rightarrow f(x) \in \,]L - \varepsilon, L + \varepsilon[$

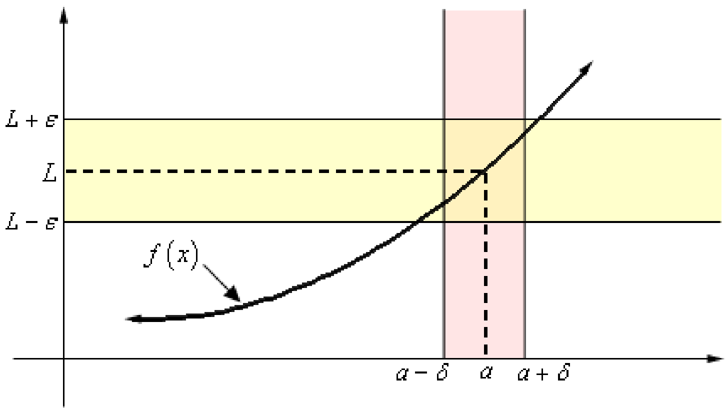


Figure 1: Geometrical representation of epsilon –delta definition of limit

This definiton of limit requires to handle with of inequality , abslute value,quantifiers $\forall, \exists$ and understanding the relationship between epslon and delta. For example show that $\lim_{x\mapsto 2}(5x - 4) = 6$. To answer this question ,

we say $\forall\varepsilon > 0, \exists\delta > 0 : \; 0 < |(5x - 4) - 6| < \varepsilon$ whenever $0 < |x - 2| < \delta$.

we need to find $\delta > 0$ so that $|(5x - 4) - 6| = |5x - 10|$

$5|x-2| < \varepsilon \Rightarrow |x-2| < \frac{\varepsilon}{5}$ this leads us to choose $\delta = \frac{\varepsilon}{5}$. Assume $0 < |x-2| < \delta = \frac{\varepsilon}{5}$.

We get $|(5x-4)-6| = |5x-10|$

$=5|x-2| < 5\left(\frac{\varepsilon}{5}\right) = \varepsilon$

So we have shown that : $0 < |(5x-4)-6| < \varepsilon$ whenever $0 < |x-2| < \delta$.

Definition of Limits of function using Heine

A function $f(x)$ has a limit L at $x = x_0$, if for every sequence $\{x_n\}$ which has a limit at x_0, the sequence $f(x_n)$ has a limit L. i.e. $\lim_{x \to x_0} f(x_n) = L$

In calculus, there is a theorem showing the equivalence between Cauchy's and Heine's definitions on limit(Mahmudov, 2013).

We can use the same example, to compare Cauchy and Heine approach on the definition of limit, show that $\lim_{x \mapsto 2} 5x - 4 = 6$. By using Heine approach , we look the sequence which are near to 2 and we find its convergence.

x_n	1.9	1.99	1.999	1.9999	2	2.01	2.001	2.0001	2.0001
$f(x_n)$	5.5	5.95	5.995	5.99995	6	6.05	6.005	6.0005	6.00005

The squence $f(x_n)$ converges to 6. Therefore, by Heine theorem $\lim_{x \mapsto 2} (5x-4) = 6$.

Both approach are the same but the level of conceptual undertsatnding of the secondary schools students are different. Some functions like logarthimic and trigonometrics functions are complicated to find the limits using Cauchy.

In addition Cauchy approach, requires students to manipulate inequalities and quantifiers wich is not easy for high school students and having some knowledge related to the topology. Furthermore, Heine approach is useful and easy for the level of high school students because it deals with the sequence and its convergence and their study sequence from primary school. Consequently, it involves students making choices and applying their understanding actively.

Conceptual understanding of Limits and continuity

The core of a scientist's work is conceptualization. It consists of organizing observations and experiences into patterns and categories that can be understood by students. An abstract representation describes complex relationships among subordinate concepts, while a simple label describes the concrete entity. Mathematical conceptual understanding involves understanding the underlying and foundational concepts behind the algorithms performed in mathematics.

Researchers claimed that students have difficulties on conceptual understanding in Limits and Continuity of functions. Intelligence and the ability to learn mathematics may not be the problem. It's probably the wrong study method that causes lack of mathematical competency, which hampers solving problems requiring mathematical skills like conceptual understanding or method which creates misconception, errors and alternative understanding of the mathematical concepts (Andamon & Tan, 2018). However, despite the fact that many contemporary accounts of mathematical understanding incorporate systematic models of conceptual understanding, there are few reports on the validation of assessment strategies derived from such models at the moment. The most common achievement tests provide only indirect and highly limited information about students' understanding of specific concepts(Kowiyah et al., 2019).

It is important to identify and correct errors and alternative understandings as part of students' difficulties in higher education by proving the activities which involve representations, problem solutions, justifications, and explanations. Teaching methods should make procedures and principles easier to learn and understand in order to effectively develop conceptual understanding.

In order to understand students' cognitive difficulties in view of the active knowledge of calculus, documenting difficulties is essential. Even though they are related, there is a difference between errors and misconceptions.

Misconceptions and errors in limits and continuity

Misconceptions are incorrect understandings of concepts, theories, or formulas. Additionally, a misconception in mathematics is a concept used when learners' conceptions do not align with the accepted meanings (Fujii, 2020). Mistakes, omissions, blunders, or inaccuracies are errors which are incorrect applications or executions of concepts, theories, or formulas (Mohyuddin & Khalil, 2016). It has been demonstrated in literature that people are unable to differentiate between misconceptions and errors. But some researchers mentioned that errors are subsets of misconceptions, all misconceptions are defined as errors, but not all errors are misconceptions. Misconceptions result in errors, or misconceptions are types of perceptions that produce systematic errors (Mishra, 2020).

Many students experience systematic errors over a long period of time, and since the problems can be solved with current knowledge and tools, it is relatively simple and easy to research them. Several factors could contribute to systematic errors, including student knowledge of procedures, conceptual knowledge, or links between these two types of knowledge. Errors are generally the result of misconceptions.

Many researchers prefer to use alternative understanding instead of using misconceptions because it is easier to correct students' alternative understanding than correct misconceptions(Mishra, 2020; Waluyo et al., 2019). It is possible to recognize errors in learners' artifacts, such as text or dialogue, written by them. Sometimes misconceptions can even be hidden in right answers, but they are often concealed from the undiscerning observer (Mohyuddin & Khalil, 2016). Students may struggle to grasp fundamental concepts covered in class, despite the best efforts of their teachers. There are some students who give the right answer but use incorrectly memorized words. The students' failure to understand the underlying concepts becomes evident when they are questioned more closely. It is common for students to solve numerical problems by using algorithms without understanding the underlying scientific theory. Students should be shown the reasoning processes that lead to algorithms as well as how concepts are generalized. Most students, however, don't grasp fundamental concepts from the beginning of their studies, and this shortcoming can hinder progress (Devard, 2022).

Several researchers examined the understanding of limits by first-year students and found that students focused solely on manipulative aspects and ignored the concept of limits. Students often use the concept of limit not being defined when it is a function that cannot be defined at that point. Moreover, they confirm that students think that limits are simply found by substituting the limit value into the expression. It was also found that students consider limits to be similar to function values. Studies show that students who have a gap in graph interpretation, algebraic manipulations, and asymptotes have difficulties with conceptions of limit continuity (Areaya & Sidelil, 2012; Denbel, 2014).

Previous studies mentioned that there is a deficiency of content and pedagogical knowledge among many teachers regarding the concept of limit and continuity. Teachers often struggle to understand the meaning of inequality and quantifiers in complex statements such as those related to continuity and limit definitions. A formal definition of limit and continuity is difficult for most teachers to convey verbally in the correct way. In addition, many teachers focus on the algebraic approach, which leads students to ignore other methods of understanding the concept of limit and continuity (Donmez & Basturk, 2010; Mastorides & Zachariades, 2004).

Theoretical framework

This study is underpinned by Piaget's theory of constructivism which ~~is~~ emphasizes that new objects and events should be compared intellectually to earlier experiences. Learners should build their mathematical knowledge based on their existing knowledge. Therefore, it is important for mathematics teachers to create a link between existing mathematical knowledge and new mathematical knowledge. It has been found that teachers and their students benefit in several ways when Piaget's theory is used in the classroom. Teachers gain more insight into their students' thinking by using Piaget's theory in the classroom (Beilin & Pufall, 2013).

For thinking capacity, Piaget described figurative and operative aspects, which are different but complementary. Learners use the operative thought to see what will happen next based on what has already occurred, and this requires intellectual abilities. Figurative representations involve imitations, perceptions, and mental images. The example below illustrates the two concepts.

Discuss the continuity of $f(x) = \frac{x^3 - 5x^2 + x - 5}{x-5}$; In this case students should know the continuity concept. i.e. $\lim_{x \to 5} f(x) = f(5)$ this is Figurative representation.

$\lim_{x \to 5} \frac{x^3 - 5x^2 + x - 5}{x-5} = \frac{0}{0}$, in this case learners should be able to remove this indeterminate form by using factorization or Hospital rule (Figurative). Then apply rule

$\text{Lim}_{x \to 5} \frac{x^3 - 5x^2 + x - 5}{x-5} = \lim_{x \to 5} \frac{(x^2+1)(x-5)}{x-5}$ (Figurative representation)

$\lim_{x \to 5}(x^2 + 1) = 26$. (This is an Operative representation).

Or $\lim_{x \to 5} \frac{(x^3 - 5x^2 + x - 5)'}{(x-5)'}$ (Figurative representation)

$$\lim_{x \to 5} 3x^2 - 10x + 1 = 26 \text{ (Operative representation)}$$

Finding $f(5) =$ not defined (Operative). The conclusion that $\lim_{x \to 5} f(x) \neq f(5)$then $f(x)$ is discontinuous at x=5 (Figurative representation).

As a result of Piaget's theory of constructivism, teachers are required to create a curriculum plan that enhances the conceptual and logical progress of their students. Experiences-or connections to the surrounding environment play a crucial role in the education of students, so teachers need to emphasize this importance.

Methodology

Research Approach

This research employed the explanatory sequential mixed-method approach, which entailed the collection and analysis of the quantitative data, followed by the collection and analysis of the qualitative data. Findings from the two types of data were then integrated to arrive at a grand finding for better understanding of the phenomenon being studied. Quantitative data was use to collect, analyze and interpret the marks and identifying the errors while qualitative data was used to collect the type of misconception and errors students faced during learning limits and continuity of functions.

Sampling

This research was conducted during the3rd term academic year 2021/2022 based on Rwandan school calendar. It was conducted at Gicumbi and Burera Districts in Northern Province. 252 senior four students who had mathematics in their options and 21 teachers who were teaching mathematics from senior four to senior six were selected purposively from nine public schools.

The characteristics of students and teachers are described in Tables 1 and 2 respectively.

Table 1: Students' characteristics (n = 252)

Characteristics	Category	Frequency	Percentage
Gender	Female	122	48.4
	Male	130	51.6
District	Gicumbi	143	56.7
	Burera	109	43.3

Table 2: Teachers' characteristics (n = 21)

Characteristics	Category	Frequency	Percentage
Gender	Female	5	38.2
	Male	16	61.8
Qualifications	Bachelor	15	71.43
	Masters	5	23.81
	Postgraduate	1	4.76
Teaching experience	0 – 2years	8	38.1

3- 5years	7	33.3
Above 5 years	6	28.6

Instruments

Instruments used to collect data were the Limit and Functions Achievement Test (LFAT), the Reformed Teaching Observation Protocol (RTOP) and two focus group interview guides; ~~one~~ each for students and teachers. The focus group interview items were informed by the outcome of the analysis of the quantitative phase of the study.

Questionnaire for achievements test was designed to find out how students conceptualize limits and continuity of functions from different perspective while content-based interviews were conducted to provide explanation for the observations in the quantitative phase of the study, which enriches the study findings. In addition, the researchers used reformed teaching observation protocol (RTOP) to see how teachers Pedagogical content knowledge relationship may affect student's conceptual understanding of limits and continuity of functions.

To ensure validity and reliability all instruments were piloted and were checked by four calculus university lecturers and three secondary school mathematics teachers. Cronbach Alpha determined that the instrument had a reliability coefficient of 0.75.

Category of questions in achievement and diagnostics assessment are described in Tables 3.

Table 3: Category of questions

Category	Questions
Category 1	Finding limits using definition
Category 2	Concept of limit and continuity
Category 3	Limit and continuity for piecewise function
Category 4	Limit and continuity for rational function (Removable discontinuity)
Category 5	Sketching graph and decide the region where the function is continuous
Category 6	Limits of point from graph and discuss its continuity
Category 7	Real word problems related to the type of discontinuity
Category 8	Algebraic method for finding limits of functions.

Phases of Data Collection

During the phase one, the diagnostic test was written by 252 students from nine schools that are in senior four with mathematics in their options. Students' prior knowledge of Limits and Continuity were tested after validation of the LFAT.

In the second phase, nine teachers representing nine schools were observed during classroom observation using RTOP. LFAT which has eight items was also administered to 252 students during normal period of class. Each student had a unique assigned identity/code which was determined by the position of their names in the official class roster.

During the third phase, the interview was conducted on selected 30 students among 252 students who showed more alternative understanding or misconceptions from their answer sheet. Also, nine teachers who were observed during classroom observation period and 12 teachers who were not observed but experienced in teaching mathematics in senior four, five and six were interviewed in four weeks after marking LFAT. The interviews lasted fifteen and 30 minutes for students and teachers respectively. Each student was interviewed in a quarter hour while each teacher was interviewed in a half hour. The interview was audio recorded and notes were taken of non-verbal gestures. An audio recording of each interview was used. Interviews provided an opportunity to identify both meaningful and erroneous conceptions. Table 4 describes in summary the phase of the study, instruments and data analysis methods used

Table 4: Summary of the phases of the study

Phase	Method	Instrument	Analysis
One	Diagnostic assessment with 252 students	Diagnostic test	Descriptive statistics.
Two	Teaching and lesson observation for 9 teachers Achievement test with 252 students	RTOP instrument Achievement test	Qualitative analysis Descriptive statistics.
Three	Interviews with 100 students and 21 teachers	Interview guide	Qualitative analysis

Data Analysis

Quantitative data were analyzed using SPSS version 25(IBM SPSS STS25) by computing frequencies (counts and percentages) while qualitative data were analyzed using Content analysis which examines the presence, meanings, and relationships of certain concepts. There can be a variety of outcomes, interpretations, and meanings associated with each type of analysis.

RESULTS AND DISCUSSIONS

In this section, the results of the study are presented and discussed. Table 5 shows that students at the selected schools have difficulties or performance and common errors in learning limits and continuity of functions. The findings of the data analysis revealed the common errors, some factors that influence student's misconception, type of errors with some examples from category of questions students faced difficulties in learning limits and continuity of functions. Type of errors and its difficulties are represented in table 6. Representation of the results were set out on the percentages for each item.

Table 5: Result on learning difficulties presented by students in limits and continuity concepts.

Items	MARKS %									
	0		25		50		75		100	
	N	%	N	%	N	%	N	%	N	%
Finding limit using Definition	235	93.3	11	4.4	2	0.8	3	1.2	1	0.4
Concept of limit and continuity	18	7.1	30	11.9	111	44.0	46	18.3	47	18.7
Limit and continuity of piecewise function	38	15.1	33	13	61	24.2	45	17.9	75	29.8
Limit and continuity for rational function(removable discontinuity)	83	32.94	0	0	71	28.2	0	0	98	38.9
Sketching graph and decide the region or point where function is discontinuous	98	38.9	23	9.1	26	10.3	37	14.7	68	27.0
Limit of point from a graph and discuss it's continuity	86	34.1	59	23.4	63	25.0	24	9.5	20	7.9
Finding limit using Algebraic method	0	0	8	3.2	37	14.7	65	25.8	142	56.3
Real word problem related to the type of discontinuity	35	13.9	55	21.8	56	22.2	27	10.7	79	31.3

From Table 5, it was observed that 235 (93.3%) scored an average of 0.00 % which shows that there is a huge challenge in finding limits by using definition. Students have a difficulty on understanding the definition, according to many teachers. Figure 1shows a sample from the answer sheet where a student used algebraic approach instead of using a definition.

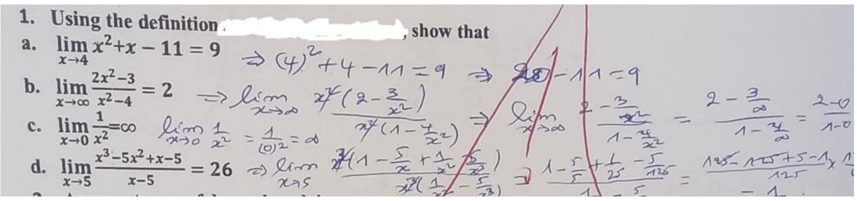


Figure2: A student's demonstration of a challenge in using definition to find limits.

During classroom observation, there were several questions that students asked regarding notations, connections between the formal approach and the informal approach, and the usefulness of formal definitions. One student asked the question "Where do ε and δcome from?" When reading through the definition, other students inquired if there was a relationship between ε and δ, L and a , and y and x. Additionally, they believed ε, L, and y were related, but did not know how, and δ, a, and x were related, but again, they didn't know how. It was noted that students did not understand how to interpret the absolute value and inequalities of Cauchy approach of the definition of limits algebraically and geometrically. Additionally, Teachers have problem of handling inequalities and related ε to δ.

Majority of the studies claimed that students have difficulty in understanding Cauchy definition of limits when learning calculus, as evidenced in previous research(Mahmudov, 2013). For students, Cauchy approaches are generally not motivating, since they are used only for problems that can be solved more intuitively (Fernández, 2004; Seager, 2020).

It was observed that on the category of concept of limit and continuity most of the respondents answered well the definition of existence of limit and continuity. Only 18 out of 252 (7.1%) students scored an average of 0.00% and 30 out of 252 (11.9%) students scored average of 25% marks,111(44%) scored on the average of 50%, 46(18.3%) students scored average of 75% and 47(18.7) scored an average of 100%. Students have cognitive conflicting related to false arguments on the concept of limit and continuity based on the examples above.

a. The existence of limit at a point implies the continuity at that point
b. The function must be defined at that point to exist the limit of that function at a point.

Some students agreed that the function must be defined at that point to be continuous which is false because on the piecewise function, it may be defined at point but the limit does not exist which fails the function to be continuous. Cognitive conflict may arise in mathematical situations when conflicting conceptions are evoked simultaneously. In interviewing students, some questions were aimed at causing cognitive conflict to see if they are sure for their responses. A total of 30 students out of 252 who were interviewed show the confusion of the existence of limit with existence of continuity. Misunderstandings related to the relationship between a limit and continuity at a point reflects to improper mental representations. To relate the concepts mentioned above meaningfully, the student should have mature conceptions of those concepts. In addition, some teachers mentioned that it is enough for a function to be defined at the point to be continuous. This shows a gap for teachers on the understanding definition of continuity function.

This result is supported by the previous findings of researchers where they agree that there is the confusion on the formation of definition of continuity (Areaya & Sidelil, 2012; Ashlock, 2019; Bezuidenhout, 2001; Budak & Ozkan, 2022; Maharajh et al., 2008). During the observation many activities provided by teachers did not focus on all conditions of the continuity of function. Most of the questions based only on the existing limit and students lose the concentration on the existence of the image of that point on function.

The findings show that on item of limit and continuity of piecewise function,38(15.1%) students scored average of 0.00%, 33(13%) students scored average of 25%, 61 (24.2%) students scored average of 50% marks, 45(17.9%) students scored 75% and 75 (29.8%) students scored an average of 100%.

These lower performances on the limit and continuity of piecewise function are influenced by the gap of connecting dynamic-theoretical conception where according to some students, continuous functions can only be defined by one equation. Given polynomial function like $f(x) = x^2 - 3x + 3$, it is easier for many students to say that the function is continuous because every $x$has value y, but given piecewise function like:

$$f(x) = \begin{cases} 2x + 1, & x > 5 \\ 3x - 4, & x < 5 \end{cases}$$

Many students mentioned the left and the right limits but did not Indicate whether the value x was approaching in the domain, this reflects on the decision of continuity or discontinuity, but some were challenged with finding the limit from left or right of a point. Instead of finding the limit, they solve each subdomain.

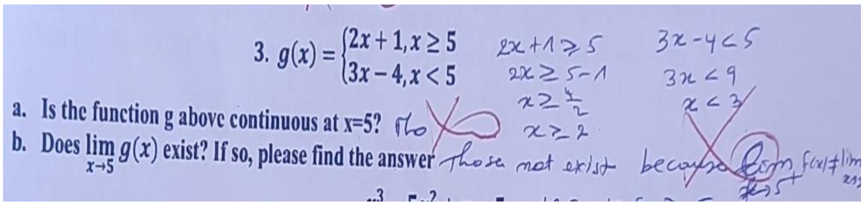


Figure 3: example of answer of student who ~~failure~~ to find limit of point for piecewise function

During interview they mentioned that it was not a single function, but comprised two. Their responses show that there is a gap of the knowledge of sketching piecewise function and finding the domain. According to different studies, students who don't understand the concept of the domain function may have difficulties understanding piecewise functions (Triutami et al., 2021). Piecewise functions are continuous on a given interval in their domain if their sub functions are continuous along their respective intervals (subdomains), and the boundaries between the subdomains are continuous.

Based on the Table 5, on the item of limit and continuity for rational function (removable discontinuity), 83(32.9%) students scored on average 0.00%, 71(28.2%) students scored on average of 50% and 98(38.9%) students scored an average of 100%. Unsatisfactory result on this item shows that continuity of rational functions, asymptotes and limits and image of functions rational function are not connected appropriately.

Given $f(x) = \frac{x}{x-1}$ it is easier for students to determine that this function is not continuous at $x = 1$ because it is outside the domain and $f(1)=\infty$. But with $f(x) = \frac{x^3-5x^2+x-5}{x-5}$, it may not be easy for them because they may simplify the function $f(x) = x^2 + 1$ and write $f(5) = 26$ which is wrong. Figure 3 shows some confusions on continuity of rational functions,

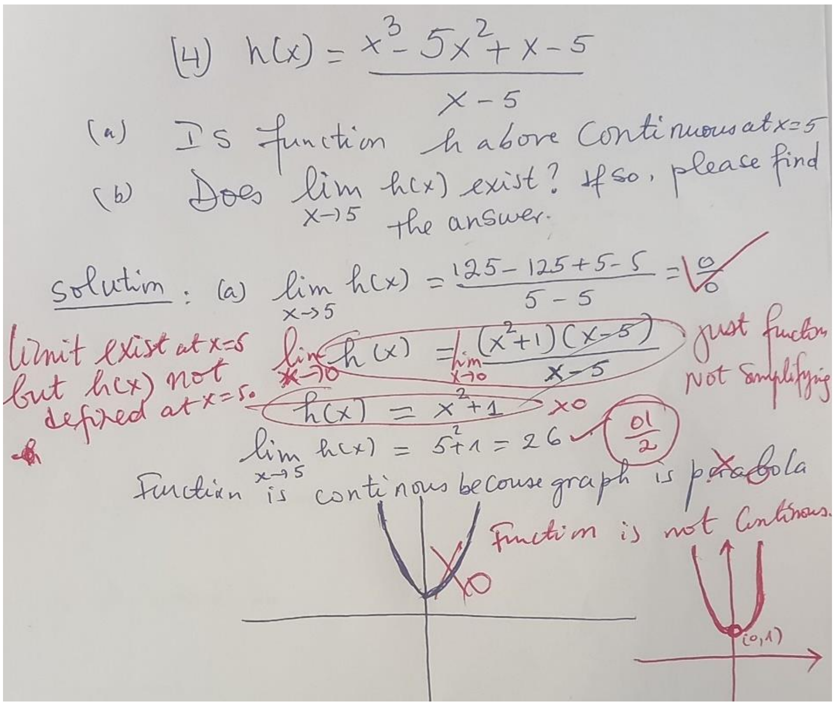


Figure 4: Example of student's written work on item of Limit and continuity for rational function

Misconceptions about functions and limits have been extensively studied by researchers. Few studies have been conducted on rational functions, their asymptotes, and the links between asymptotes, limits, and continuity (Mrdja et al., 2015; Nair, 2010).It observed that some students are not able to factorize rational function for removing the indeterminate form of $\frac{0}{0}$. Most of the students immediately apply Hospital rule which implies that in case you give them rational function without having the knowledge on derivative, majority of students may fail to remove indeterminate form of $\frac{0}{0}$.

Table 5, shows that on the item that has to do with sketching graph and determining the region or point of discontinuity of the function the majority of students have difficulties on sketching some functions like piecewise, absolute value and trigonometric functions and show the region where it is continuous. It was observed that 98(38.9%) scored an average 0.00%, 23(9.1%) scored an average of 25%, 26 (10.3%) scored an average of 50%, 37(14.7%) scored an average 75% while 68(27%) scored an average of 100%. It was observed that there was a mismatching of domain of the definition and the continuity of a function. During the interview students explained how to ~~drew~~ the functions like

$f(x) = x^2$ they remembered that the bottom of this parabola is curved upward, also for the

$f(x) = x^2 - 3x + 3$ ~~they~~ they guessed right that it is curved upward but were not sure at which points it is turned on. For irrational function, many students solve or simplified the functions instead of sketching which shows the gap in algebra to differentiate function and equation.

It was also observed that given trigonometric function like $f(x) = sinx$ or $f(x) = cosx$ students replied that its graph will be sinusoidal but if you change the function $f(x) = \tan x$ it is not easy for them to sketch. During the teaching, many teachers focused on rough sketching method which was memory-based graphics sketching rather than pointwise approach in drawing a graph. Many students mentioned that pointwise method takes a long step to sketch function. It was observed that students who were not able to determine the domain of piecewise function had the wrong graphs. Figure 4 shows some demonstrations from the students.

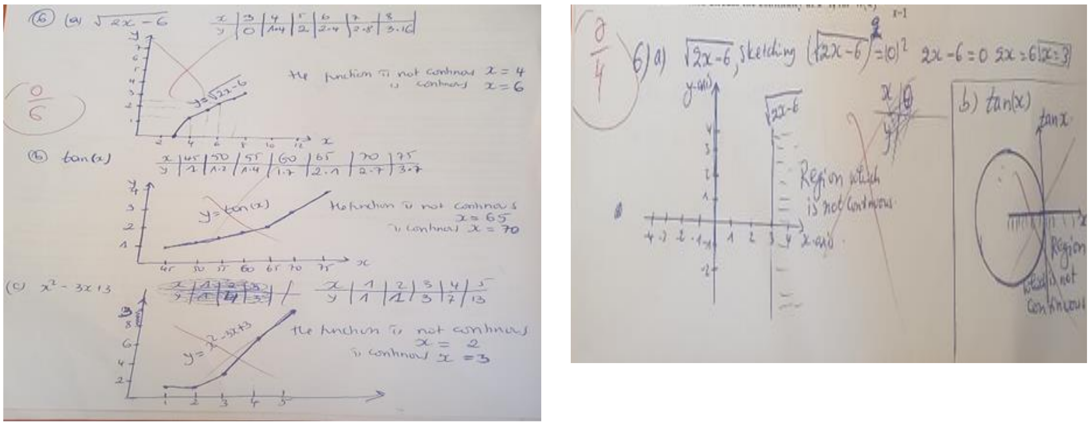


Figure 5: Example of student's written work who failed to sketch functions and interpret its continuity or not.

Based on the results in Table 5, it was observed that on the item of finding limits of points from a graph and discussing its continuity 86 (34.1%) scored an average of 0.00%, 59(23.4%) scored an average of 25%, 63(25%) scored an average of 50% and 24(9.5%) scored an average of 75% and 20 (7.9 %) scored an average of 100%. There is misconception on connecting the image and the limit of functions at a point. As seen in figure 6, many students mentioned that $\lim_{x\to a} f(x) = f(a)$ but this is not always true.

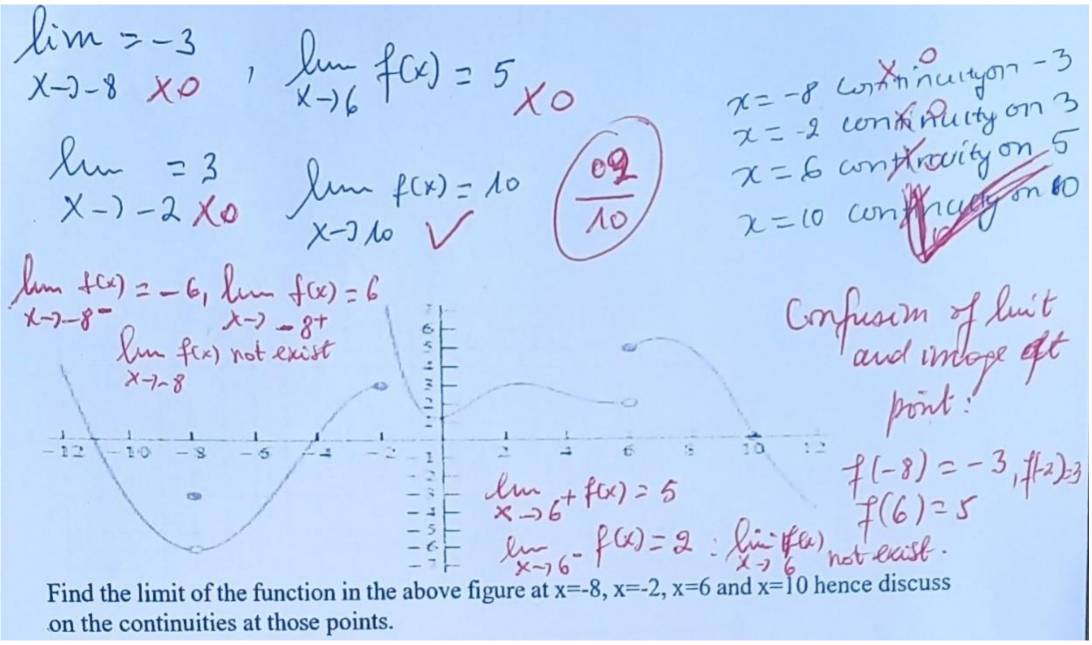
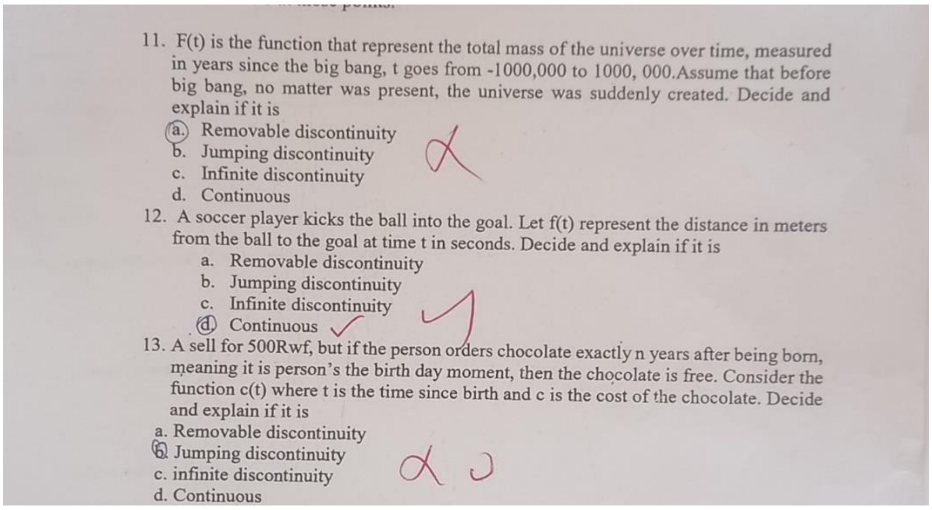


Figure 6: An example of a student's written work an item to find limit from a graph and interpret if it continuous at point or not.

It will be true when function is continuous at $x = a$. Looking at the answer sheet it is observed that ~~we~~ many students gave the correct responses on the point of discontinuity from graph even if they missed evaluating the limit from the graph which shows that analytical approaches to semiotics are lacking when compared to graphical representations. During interview, students were able to decide the continuity or discontinuity of point from graph but not able to find the limits of or image of that points. It was observed that there was a gap on algebraic presentation from graphical interpretations.

On the item of finding limit using the algebraic method the findings on table5, shows that only 142(56.3%) students scored an average 100%, 65(25.8%) scored an average of 75%, 37(14.7%) students scored an average of 50% and 8(3.2%) scored an average of 25%. This good performance is supported by many algebraic exercises included in calculus textbooks strongly promote the algebraic approach that results from representing functions as formulas. In certain situations, students may disregard the benefits of other approaches if they are taught to use algebraic principles only.

Table 5 shows that on item real word problem related to the type of discontinuity 79(31.3 %) scored an average of 0.00%, 55(21.8%) scored an average of 25%, 56 (22.2%) scored an average of 50% ,35(13.9%) scored an average 75% and 27(10.7%) scored an average 100% and it shows that majority of the students did not understand the concept of continuity or discontinuity in word problems. During the interview many students mentioned that teachers did not give more examples on real word problems related to the continuity or discontinuity. Some teachers were limited in their knowledge of real-life applications of continuity and functions. This picture shows some response of the learners



11. F(t) is the function that represent the total mass of the universe over time, measured in years since the big bang, t goes from -1000,000 to 1000, 000.Assume that before big bang, no matter was present, the universe was suddenly created. Decide and explain if it is
   a. Removable discontinuity
   b. Jumping discontinuity
   c. Infinite discontinuity
   d. Continuous
12. A soccer player kicks the ball into the goal. Let f(t) represent the distance in meters from the ball to the goal at time t in seconds. Decide and explain if it is
   a. Removable discontinuity
   b. Jumping discontinuity
   c. Infinite discontinuity
   d. Continuous
13. A sell for 500Rwf, but if the person orders chocolate exactly n years after being born, meaning it is person's the birth day moment, then the chocolate is free. Consider the function c(t) where t is the time since birth and c is the cost of the chocolate. Decide and explain if it is
   a. Removable discontinuity
   b. Jumping discontinuity
   c. infinite discontinuity
   d. Continuous

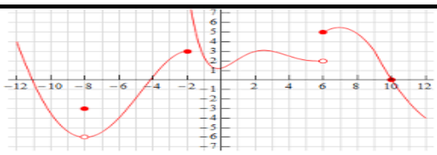
Figure 7: The written work of a student who could not determine whether the given real-world problem is continuous or discontinuous.

Teaching mathematics using real-world connections is emphasized in mathematics-education community. Moreover, researchers claim that learners who studied in integrated learning situation are additionally expected to apply their learning in various contexts. Also, studies show that when students are faced with problematic items, they exclude real-world knowledge and realistic considerations (Gainsburg, 2008; Stacey, 2015). Experimental classroom setting which exclude real-world knowledge can create belief among these students that school arithmetic word problems are tricky.

Furthermore, based on the result from the achievement test and interview researchers observed four types of errors and difficulties in learning limits and continuities as shown in Table 6

Table 6: Student's misconceptions and errors in limits and continuity of functions

Items	Some examples	Common errors	Type of errors
Finding limits using Definition	Show that $\lim_{x\to 5}\frac{x^3-5x^2+x-5}{x-5}=26$	All students use the algebraic method instead of using the definition of limit	Conceptual errors where there is a deficiency in understanding of the concept
Limit and continuity of piecewise function	Discuss the continuity at x=5 $g(x)=\begin{cases}2x+1, x\geq 5\\3x-4, x<5\end{cases}$ $f(x)=\begin{cases}4x-3, & x\neq 5\\15, & x=5\end{cases}$	38 students are not able to identify the limit from the left and right $g(x)$ of at x=5. 52 students are able to find the $\lim_{x\to 5} f(x)$ but are not able to connect to the domain of $f(x)$	Procedural errors where steps are not conducted properly Systematics errors where there is a lack of connecting procedural to conceptual knowledge
Limit and continuity for rational function(removable discontinuity)	Discuss the continuity of $h(x)=\frac{x^3-5x^2+x-5}{x-5}$	While removing the indeterminate form $\frac{0}{0}$ 147 students applied hospital which requires skills on derivative. Skills of factorization is missing	Fractural errors or technical errors where knowledge related to factorization is insufficient
Sketching graph and decide the region or point where the function is discontinuous.	Sketch the following function and decide which region it is continuous f(x)= $\sqrt{2x-6}$	Most of students sketched $\sqrt{x}$and x=3, because of roughly method of sketching function, also many students solve the function instead sketching	Procedural errors where students are not able to do all steps properly. Also, there is conceptual errors of differentiating

			function and equation.
Limits of points from a graph and discuss its continuity	 Discuss the continuity at x=-8, x=10 and x= 6	176 students confused image of point and its limit from graph. There is errors where $\lim_{x \to a} f(x) = f(a)$	There is systematic errors lack of connecting procedural to conceptual knowledge
Real word problem related to the type of discontinuity	F(t) is the function that represent the total mass of the universe over time, measured in years since the big bang, t goes from -1000,000 to 1000, 000.Assume that before big bang, no matter was present, the universe was suddenly created. Decide and explain if it is a. Removable discontinuity b. Jumping discontinuity c. Infinite discontinuity d. Continuous	189 students are not able to decide whether it continuous or discontinuous.	Conceptual errors where there is lack of understanding the concept of continuity.

Observation Data

A study was conducted to observe how limits and continuity of functions were taught and learned in schools. For this study, we observed the classroom setting, teachers' teaching methods, and students' behavior during lessons to gain a better understanding on limits and continuity of functions. Also, we observed the factors that influence misconceptions and errors. Table 7 provide a summary of the observation checklist used throughout the classroom observations while Table 8 represent the factors influencing misconceptions.

Table 7: Observation themes and sub-themes using RTOP

Main themes	Subthemes
Lack of multiple representation	• Presentation lacking clarity and concreteness • Focus on one presentation (algebraic approach)

Lack of cooperative learning	• A variety of means and media were less often used by students to communicate their ideas.
Lack of incident of creative and critical thinking	• Conceptual understanding is ~~strongly~~ promoted by the lesson ~~less~~ • It was encouraged to use abstraction (e.g., symbolic representation, theory building) when appropriate.
Lack of active learning	• Students were reflective about their learning. • It was highly valued to have intellectual rigor, constructive criticism, and to challenge ideas.

Table 8: Factors influencing students' misconceptions and errors in learning limits and continuity of functions

Source of errors	Explanations
Superficial Understanding	• Students focus on the memorization of the rules without understanding
Instructional Method	• Lesson design may create the misconceptions in case teacher is not able to match pedagogical knowledge and content knowledge • Do not give sufficient examples • Lack of different method of presentation of the concepts
Insufficient knowledge of the basic concept	• Basic knowledge of factorization • Basic knowledge on domain of function
Carelessness	• Student do not reflect to their answers, no verification or checking if their answers make

	sense.
External Limitation	• Too much syllabus to complete • Different students’ abilities • Large number of students • School activities,..

CONCLUSIONS AND RECOMMENDATIONS

This paper was about exploring students’ conceptual understanding of limits and functions. It highlighted some learning difficulties, errors and misconceptions that students faced in learning limits and continuities. The findings showed that most students from Burera and Gicumbi Districts have difficulties in finding limits using definition, sketching functions and locating the continuous region or points, finding limits of points from graph and determining whether a function is continuous at a particular point or not. Most of students have difficulties on understanding limits and continuities applied to real-worlds situations, as they could not recognize whether the functions are continuous or discontinuous. The study also found that students held misconceptions and errors in limits and continuity of functions ~~such~~ limited skills of the relation between limit and continuity. Also, they rely largely on isolated facts and procedures. Consequently, this type of situation is largely caused by learning and teaching approaches that rely mostly on procedural aspects of calculus while neglecting conceptual underpinnings. Several teachers used the books that include stereotyped exercises that encourage the use of computational approaches over relational understandings.

It is essential to teach students limits and continuity along with manipulative skills to develop conceptual knowledge. The teaching of limit and continuity must be designed to facilitate this type of progress where students should be taught to understand conceptually, as well as manipulate, mathematical content. A teaching approach and mathematical tasks should be structured so that students understand and experience the meaning of mathematical symbols, the importance of interpreting symbols and using them to represent concepts, and the fact that the ability to manipulate algebraic expressions to find limits and continuity does not imply understanding of the symbol meanings.

Learners can develop concepts from intuitive to analytical by explaining limits and continuity using graphs, algebraic methods, tables of values, and verbal descriptions of functions. Additionally, the study suggests that teachers should not ignore students' misconceptions. Also, teachers should pay attention to the learners' skills and knowledge when teaching mathematical concepts. Students should be helped to discover concepts for themselves rather than teaching them everything that they need to learn.

As many students performed well on continuity from graph, we recommend researchers to emphasize on studying how technology might affect students' understanding of limits and continuity, such as computer programs and graphing calculators (Dynamical mathematical software). Derivative, asymptote and, convergence of improper integrals and series are part of the application of limit and continuity. We are encouraging other researchers to investigate the effect of student's errors and alternative understanding toward the student's performance on derivative, asymptote and, converge improper integrals and series. Also, based on the unsatisfactory performance on the questions related to the finding limits by definition we recommend the teachers to focus on the approach of Heine which is easier to understand for the level of high school students than Cauchy approach.

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